Estimates for Schrödinger operators on modulation spaces (Harmonic Analysis and Nonlinear Partial Differential Equations)

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Estimates for Schrödinger operators on modulation spaces

By

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Abstract

The purpose of this article is to give a brief survey and some remarks on estimates on modulation spaces for Schrödinger evolution operators with quadratic and sub-quadratic potentials, which were obtained by authors in [14], [15], [16], [17].

§ 1. Introduction

In this article, we shall consider the initial value problems for the time dependent Schrödinger equation

$$
\begin{cases} i\partial_t u(t,x) = -\frac{1}{2}\Delta u(t,x) + V(t,x)u(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x\in\mathbb{R}^n \end{cases} \tag{1.1}
$$

in the framework of modulation spaces $M^{p,q}(\mathbb{R}^n)$. Here $i=\sqrt{-1}$, $u(t,x)$ is a complex valued function of $(t,x)\in\mathbb{R}\times\mathbb{R}^n$, $V(t,x)$ is a real valued function of $(t,x)\in\mathbb{R}\times\mathbb{R}^n$, $u_0(x)$ is a complex valued function of $x\in\mathbb{R}^n$, $\partial_t u = \partial u/\partial t$ and $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$.

Here and in what follows, we assume that $V\in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ and for all multi-indices α with $|\alpha|\ge 2$ (or $|\alpha|\ge 1$) there exists $C_\alpha>0$ such that

$$
|\partial_x^\alpha V(t,x)| \le C_\alpha, \quad (t,x)\in\mathbb{R}\times\mathbb{R}^n. \tag{1.2}
$$

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Before going into this problem, we recall that the Schrödinger operator of a free particle $e^{\frac{1}{2}it\Delta}$ doesn't preserve L^p norm. More precisely, if $V(t,x)=0$, then

$$\sup_{u_0\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}} \frac{\|u(t,\cdot)\|_{L^p}}{\|u_0\|_{L^p}} < \infty$$

if and only if $p=2$ (if $t\neq 0$), where $u(t,x)$ is the solution of (1.1) with $u(0,x)=u_0(x)$. Moreover, for $2\le p\le\infty$ with $1/p+1/p'=1$, the following inequality holds.

$$\|u(t,\cdot)\|_{L^p} \le C|t|^{-n(1/2-1/p)}\|u_0\|_{L^{p'}}, \quad u_0\in L^{p'}(\mathbb{R}^n)$$

(see for example Linares-Ponce [19]).

In the last decade, corresponding estimates for $M^{p,q}(\mathbb{R}^n)$ were obtained by Bényi-Gröchenig-Okoudjou-Rogers [1] and Wang-Hudzik [27] (see also Toft [24] and Wang-Zhao-Guo [28]).

Theorem A (Bényi-Gröchenig-Okoudjou-Rogers)**.** Let $1\le p,q\le\infty$ and $\varphi_0\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$. Suppose $V(t,x)=0$. Then there exists $C>0$ such that

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi_0}} \le C(1+|t|)^{n/2}\|u_0\|_{M^{p,q}_{\varphi_0}}, \quad u_0\in\mathcal{S}(\mathbb{R}^n)$$

for all $t\in\mathbb{R}$, where $u(t,x)$ is the solution of (1.1) with $u(0,x)=u_0(x)$.

Theorem B (Wang-Hudzik)**.** Let $2\le p\le\infty$, $1\le q\le\infty$, $1/p+1/p'=1$ and $\varphi_0\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$. Suppose $V(t,x)=0$. Then there exist C and $C'>0$ such that

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi_0}} \le C(1+|t|)^{-n(1/2-1/p)}\|u_0\|_{M^{p',q}_{\varphi_0}}, \quad u_0\in\mathcal{S}(\mathbb{R}^n)$$

and

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi_0}} \le C'(1+|t|)^{n(1/2-1/p)}\|u_0\|_{M^{p,q}_{\varphi_0}}, \quad u_0\in\mathcal{S}(\mathbb{R}^n)$$

for all $t\in\mathbb{R}$, where $u(t,x)$ is the solution of (1.1) with $u(0,x)=u_0(x)$.

These estimates show that $e^{\frac{1}{2}it\Delta}$ preserves $M^{p,q}$ norm for all p and q. Motivated by Theorems A and B, we have obtained the following estimates ([14], [15], [16]).

Theorem 1.1 (Kato-Kobayashi-Ito)**.** *Let $1\le p,q\le\infty$ and $\varphi_0\in\mathcal{S}(\mathbb{R}^n)\setminus\{0\}$. Suppose $V(t,x)=0$. Then*

$$\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} = \|u_0\|_{M^{p,q}_{\varphi_0}}, \quad u_0\in\mathcal{S}(\mathbb{R}^n) \tag{1.3}$$

holds for all $t\in\mathbb{R}$, where $u(t,x)$ and $\varphi(t,x)$ denote the solutions of (1.1) *with $u(0,x)=u_0(x)$ and $\varphi(0,x)=\varphi_0(x)$.*

Remark 1. We note that the norms on the left-hand and right-hand sides of (1.3) are measured by different windows. More precisely, we have

$$|W_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)| = |W_{\varphi_0}u_0(x(0),\xi(0))|. \tag{1.4}$$

Here $W_{\varphi(t,\cdot)}(u(t,\cdot))$ and $W_{\varphi_0}u_0$ denote the wave packet transforms of $u(t,\cdot)$ and $u_0(\cdot)$ with respect to $\varphi(t,\cdot)$ and $\varphi_0(\cdot)$, respectively (see Section 2.1 for the definition). And $x(s), \xi(s) : \mathbb{R} \to \mathbb{R}^n$ are solutions of

$$\frac{d}{ds}x(s) = \xi(s), \quad \frac{d}{ds}\xi(s) = 0, \quad \textit{with} \quad x(t) = x, \quad \xi(t) = \xi$$

(i.e., $x(0) = x - \xi t$ and $\xi(0) = \xi$).

Remark 2. We can easily recover Theorems A and B from (1.4) (see [14] for details).

Theorem 1.2 (Kato-Kobayashi-Ito)**.** *Let $1 \le p \le \infty$ and $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Suppose $V(t,x) = \pm\frac{1}{2}|x|^2$. Then*

$$\|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} = \|u_0\|_{M^{p,p}_{\varphi_0}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n)$$

holds for all $t \in \mathbb{R}$, where $u(t,x)$ and $\varphi(t,x)$ denote the solutions of (1.1) *with $u(0,x) = u_0(x)$ and $\varphi(0,x) = \varphi_0(x)$.*

Remark 3. More precisely, we have

$$|W_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)| = |W_{\varphi_0}u_0(x(0),\xi(0))|.$$

Here, $x(s), \xi(s) : \mathbb{R} \to \mathbb{R}^n$ are solutions of

$$\frac{d}{ds}x(s) = \xi(s), \quad \frac{d}{ds}\xi(s) = \mp x(s) \quad \textit{with} \quad x(t) = x, \quad \xi(t) = \xi$$

$$\left(\textit{i.e.}, V(t,x) = \tfrac{1}{2}|x|^2 \implies \begin{cases} x(0) = x\cos t - \xi \sin t \\ \xi(0) = x \sin t + \xi \cos t \end{cases} \right).$$

In order to state our results in a precise form, we introduce the Schrödinger operator of a free particle $e^{\frac{1}{2}it\Delta}$ defined by

$$(e^{\frac{1}{2}it\Delta}f)(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}it|\xi|^2}\widehat{f}(\xi)d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Here we use the notation $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$ for the Fourier transform of f and $f^\vee(x) = (2\pi)^{-n}\widehat{f}(-x)$ for the inverse Fourier transform.

The main results are stated as follows.

Theorem 1.3. *Let $1 \leq p \leq \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ and $T > 0$. Set $\varphi(t,x) = (e^{\frac{1}{2}it\Delta}\varphi_0)(x)$. If a real-valued function $V \in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ satisfies* (1.2) *for all multi-indices α with $|\alpha| \geq 2$, then there exist positive constants c_T and C_T such that*

$$c_T\|u_0\|_{M^{p,p}_{\varphi_0}} \leq \|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} \leq C_T\|u_0\|_{M^{p,p}_{\varphi_0}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n) \tag{1.5}$$

for all $t \in [-T,T]$, where $u(t,x)$ denotes the solution of (1.1) *in $C(\mathbb{R}; L^2(\mathbb{R}^n))$ with $u(0,x) = u_0(x)$.*

Theorem 1.4. *Let $1 \leq p,q \leq \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ and $T > 0$. Set $\varphi(t,x) = (e^{\frac{1}{2}it\Delta}\varphi_0)(x)$. If a real-valued function $V \in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ satisfies* (1.2) *for all multi-indices α with $|\alpha| \geq 1$, then there exist positive constants c_T and C_T such that*

$$c_T\|u_0\|_{M^{p,q}_{\varphi_0}} \leq \|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} \leq C_T\|u_0\|_{M^{p,q}_{\varphi_0}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n) \tag{1.6}$$

for all $t \in [-T,T]$, where $u(t,x)$ denotes the solution of (1.1) *in $C(\mathbb{R}; L^2(\mathbb{R}^n))$ with $u(0,x) = u_0(x)$.*

Remark 4. The right hand sides of the inequalities (1.5) and (1.6) were already given in Kato-Kobayashi-Ito [17]. We also remark that more general inequality for the right hand side of (1.5) (with an independent proof) were obtained in Cordero-Gröchenig-Nicola-Rodino in [3] (see also Cordero-Gröchenig-Nicola-Rodino [4], Cordero-Nicola-Rodino [7]).

Remark 5. In Theorem 1.3, we cannot expect to replace the $M^{p,p}$ norm with the $M^{p,q}$ norm, because it is impossible to replace the $M^{p,p}$ norm with $M^{p,q}$ norm in Theorem 1.2 (see [16]).

Remark 6. We note that there exists a unique solution of (1.1) under the assumption (1.2) for all multi-indices α with $|\alpha| \geq 2$. More precisely, there exists a unitary operator $U(s,t)$ on $L^2(\mathbb{R}^n)$ for $s,t \in \mathbb{R}$ such that $U(t,t_0)u_0$ solves

$$\begin{cases} i\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u - Vu = 0, \\ u(t_0,x) = u_0 \in L^2, \end{cases} \tag{1.7}$$

and

$$U(s,s) = I, \quad U(s,t) = U(t,s)^{-1}, \quad U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3),$$

where I denotes the identity operator on $L^2(\mathbb{R}^n)$ (see Fujiwara [11], [12]).

There have been many works on the dispersive estimate (L^p-L^q estimate) of (1.1)

$$\|u(t,\cdot)\|_{L^\infty} \leq C|t|^{-\frac{n}{2}}\|u_0\|_{L^1},$$

which we don't treat here (see Yajima [22], Schlag [23], Yajima [29] and references therein).

Finally, we give a remark on Miyachi type estimates for Schrödinger operators (see Miyachi [20]). In Kobayashi-Sugimoto [18], the following inclusion relation between L^p-Sobolev and modulation spaces were obtained:

$$H^p_{2n(\frac{1}{p}-\frac{1}{2})}(\mathbb{R}^n) \hookrightarrow M^{p,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \tag{1.8}$$

for $1 < p \leq 2$. Here $H^p_s(\mathbb{R}^n)$ denotes the L^p-Sobolev spaces defined by the norm

$$\|f\|_{H^p_s} := \|(\langle\cdot\rangle^s \widehat{f}(\cdot))^\vee\|_{L^p} \quad \text{with} \quad \langle x\rangle = (1+|x|^2)^{1/2}.$$

D'Ancona-Nicola [9] (which treats time-independent potentials) has pointed out that $M^{p,p}$ estimates such as Theorems 1.3 and 1.4 implies

$$\|u(t,\cdot)\|_{L^p} \leq C_T \|u_0\|_{H^p_{2n(\frac{1}{p}-\frac{1}{2})}} \tag{1.9}$$

for $1 < p \leq 2$ (Miyachi type estimate). In fact, under the assumption (1.2), we have by Theorems 1.3 and 1.4

$$\|u(t,\cdot)\|_{M^{p,p}_{\varphi_0}} \leq C(1+|t|)^{\frac{n}{2}} \|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} \leq C_T \|u_0\|_{M^{p,p}_{\varphi_0}} \leq C'_T \|u_0\|_{M^{p,p}_{\varphi_0}}$$

(for the first inequality, see for example [14, p.227]). Combining with (1.8), we obtain (1.9).

§ 2. Preliminaries

The following notation will be used throughout this article. We write $\mathcal{S}(\mathbb{R}^n)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ to denote the space of tempered distributions on $\mathbb{R}^n$, i.e., the topological dual of $\mathcal{S}(\mathbb{R}^n)$. We define

$$||f||_{L^p} = \Big(\int_{\mathbb{R}^n} |f(x)|^p dx\Big)^{1/p}$$

for $1 \leq p < \infty$ and $||f||_{L^\infty} = \operatorname{ess.sup}_{x\in\mathbb{R}^n} |f(x)|$. We use the notation $I \lesssim J$ if I is bounded by a constant times J, and we denote $I \approx J$ if $I \lesssim J$ and $J \lesssim I$.

§ 2.1. Wave packet transform

We first recall the wave packet transform defined by Córdoba-Fefferman [8]. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then the wave packet transform (with respect to ϕ) $W_\phi : f(x) \mapsto W_\phi f(x,\xi)$ is defined by

$$W_\phi f(x,\xi) := \langle f(y), \phi(y-x)e^{iy\cdot\xi}\rangle = \int_{\mathbb{R}^n} f(y)\overline{\phi(y-x)}e^{-iy\cdot\xi}dy, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

We call such ϕ the window function. The adjoint operator of W_ϕ is defined as follows: Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then the adjoint operator $W_\phi^* : F(x,\xi) \mapsto W_\phi^* F(x)$ is defined by

$$W_\phi^* F(x) := \iint_{\mathbb{R}^{2n}} F(y,\xi)\phi(x-y)e^{ix\cdot\xi}dy đ\xi, \quad đ\xi = (2\pi)^{-n}d\xi$$

for a function F on $\mathbb{R}^n \times \mathbb{R}^n$.

We remark that $W_\phi f(x,\xi)$ is a function on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$W_\phi f_1(x,\xi) + W_\phi f_2(x,\xi) = W_\phi[f_1 + f_2](x,\xi)$$

and

$$W_{\phi_1} f(x,\xi) + W_{\phi_2} f(x,\xi) = W_{[\phi_1+\phi_2]} f(x,\xi)$$

for $f, f_1, f_2 \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi, \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^n)$. Moreover for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$ with $\langle \psi, \phi \rangle = \int_{\mathbb{R}^n} \psi(x)\overline{\phi(x)}dx \neq 0$, we have the inversion formula

$$\frac{1}{\langle \psi, \phi \rangle} W_\psi^* W_\phi f = f, \quad f \in \mathcal{S}'(\mathbb{R}^n)$$

([13, Corollary 11.2.7]). In particular we have

$$\|W_\phi f\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)} = \|\phi\|_{L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n).$$

§ 2.2. Modulation spaces

We recall the definition of modulation spaces $M^{p,q}(\mathbb{R}^n)$ introduced by Feichtinger [10]. Let $1 \le p, q \le \infty$, $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Then the modulation space $M_\phi^{p,q}(\mathbb{R}^n) = M^{p,q}$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that the norm

$$\|f\|_{M_\phi^{p,q}} := \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |W_\phi f(x,\xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} = \big\| \|W_\phi f(x,\xi)\|_{L^p(\mathbb{R}^n_x)} \big\|_{L^q(\mathbb{R}^n_\xi)}$$

is finite (with usual modifications if $p = \infty$ or $q = \infty$).

We collect basic properties of modulation spaces in the following lemma.

Lemma 2.1. *Let $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$. Then*

(1) *The space $M_\phi^{p,q}(\mathbb{R}^n)$ is a Banach space, whose definition is independent of the choice of ϕ. More precisely, for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, we have*

$$M_\phi^{p,q}(\mathbb{R}^n) = M_\psi^{p,q}(\mathbb{R}^n) (= M^{p,q}) \quad \textit{with} \quad \|f\|_{M_\phi^{p,q}} \approx \|f\|_{M_\psi^{p,q}}.$$

(2) *$M^{p,min\{p,p'\}}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow M^{p,max\{p,p'\}}(\mathbb{R}^n)$, where $1/p + 1/p' = 1$. In particular, we have $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.*

(3) *If* $p_1 \le p_2$ *and* $q_1 \le q_2$*, then* $M^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^n)$*.*

(4) (*Density and duality*) *If* $p, q < \infty$*, then* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $M^{p,q}(\mathbb{R}^n)$ *and*

$$(M^{p,q}(\mathbb{R}^n))' = M^{p',q'}(\mathbb{R}^n), \quad \text{with} \quad 1/p + 1/p' = 1 = 1/q + 1/q'.$$

Let us define by $\mathcal{M}^{p,q}(\mathbb{R}^n)$ *the completion of* $\mathcal{S}(\mathbb{R}^n)$ *under the norm* $\|\cdot\|_{M^{p,q}}$ (*of course, if* $p, q < \infty$*, then we have* $\mathcal{M}^{p,q} = M^{p,q}$).

(5) (*complex interpolation*) *Let* $0 < \theta < 1$ *and* $1 \le p_1, p_2, q_1, q_2 \le \infty$*. Set*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then we have

$$(\mathcal{M}^{p_1,q_1}, \mathcal{M}^{p_2,q_2})_{[\theta]} = \mathcal{M}^{p,q}.$$

We refer to [10] and [13] for more details.

§ 3. Outlines of proofs of Theorems 1.3 and 1.4

In this section, we shall briefly explain some ideas to prove Theorems 1.3 and 1.4. The proofs are completed by showing the following lemmas.

Lemma 3.1. *Let the assumption of Theorem 1.3 hold, and* $s_0 \in [-T, T]$*. Then there exists* $C_T > 0$ *such that*

$$\|u(t - s_0, \cdot)\|_{M^{p,p}_{\varphi(t-s_0,\cdot)}} \le C_T \|u(-s_0, \cdot)\|_{M^{p,p}_{\varphi(-s_0,\cdot)}} \tag{3.1}$$

for all $t \in [-T, T]$*, where* $u(t, x)$ *denotes the solution of* (1.1) *in* $C(\mathbb{R}; L^2(\mathbb{R}^n))$ *with* $u(0, x) = u_0(x)$*.*

Lemma 3.2. *Let the assumption of Theorem 1.4 hold, and* $s_0 \in [-T, T]$*. Then there exists* $C_T > 0$ *such that*

$$\|u(t - s_0, \cdot)\|_{M^{p,q}_{\varphi(t-s_0,\cdot)}} \le C_T \|u(-s_0, \cdot)\|_{M^{p,q}_{\varphi(-s_0,\cdot)}} \tag{3.2}$$

for all $t \in [-T, T]$*, where* $u(t, x)$ *denotes the solution of* (1.1) *in* $C(\mathbb{R}; L^2(\mathbb{R}^n))$ *with* $u(0, x) = u_0(x)$*.*

In fact, the inequality (3.1) with $s_0 = 0$ corresponds the right hand side of (1.5) in Theorem 1.3. Likewise, putting $s_0 = t$ and replacing t with $-t$ and in (3.1), we obtain the left hand side of (1.5). This observation is also true for the inequality (3.2).

Moreover, since Lemmas 3.1 and 3.2 are simple modifications of Theorems 1.1 and 1.2 in [17], we only give a sketch of the proof for Lemma 3.1 and Lemma 3.2.

We first note that $v(t,x) := u(t-s_0,x)$ is the solution of

$$\begin{cases} i\partial_t v(t,x) = -\frac{1}{2}\Delta v(t,x) + \widetilde{V}(t,x)v(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ v(0,x) = v_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{3.3}$$

where $\widetilde{V}(t,x) = V(t-s_0,x)$ and $v_0(x) = u(-s_0,x)$. Similarly, $\psi(t,x) := \varphi(t-s_0,x)$ is the solution of

$$\begin{cases} i\partial_t \psi(t,x) = -\frac{1}{2}\Delta \psi(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R}^n, \\ \psi(0,x) = \psi_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{3.4}$$

where $\psi_0(x) = \varphi(-s_0,x)$.

§ 3.1. Key Lemmas

To prove Lemmas 3.1 and 3.2, we use the following lemmas whose proofs are almost repetitions of arguments in [17].

Lemma 3.3. *Let $V\in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ satisfy (1.2) for all multi-indices α with $|\alpha|\ge 2$ and define $\widetilde{V}(t,x) = V(t-s_0,x)$ for $s_0\in[-T,T]$. We consider the system of ordinary differential equations*

$$\begin{cases} \dfrac{d}{ds}x(s) = \xi(s), \\ \dfrac{d}{ds}\xi(s) = -(\nabla_x\widetilde{V})(s,x(s)), \end{cases} \tag{3.5}$$

where $x(s),\xi(s):\mathbb{R}\to\mathbb{R}^n$ and

$$\nabla_x\widetilde{V}(t,x) = (\partial_{x_1}\widetilde{V}(t,x),\cdots,\partial_{x_n}\widetilde{V}(t,x)).$$

Suppose $x(s)$ and $\xi(s)$ are solutions to (3.5) with $x(t)=x$ and $\xi(t)=\xi$. If we consider the change of variables

$$X = x(s) \quad \text{and} \quad \Xi = \xi(s),$$

then

$$\det\frac{\partial(X,\Xi)}{\partial(x,\xi)} = 1$$

for all s_0, s, t, x and ξ.

Lemma 3.4. *Let $V\in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ satisfies (1.2) for all multi-indices α with $|\alpha|\ge 1$ and define $\widetilde{V}(t,x) = V(t-s_0,x)$ for $s_0\in[-T,T]$. Suppose $x(s)$ and $\xi(s)$ be solutions to (3.5) with $x(t)=x$ and $\xi(t)=\xi$. Then there exist $C_1, C_2>0$ such that*

$$\frac{1}{\langle y-x(s)\rangle} \le \frac{C_1(1+|t-s|^2)}{\langle y-x+(t-s)\xi\rangle}$$

and

$$\frac{1}{\langle \eta - \xi(s) \rangle} \leq \frac{C_2(1+|t-s|)}{\langle \eta - \xi \rangle},$$

for all $s_0.s, t, x$ *and* ξ.

§ 3.2. Proof of Lemma 3.1

We only consider the case $t \in [0,T]$. We can treat the case $t \in [-T,0]$ in the same way. The main idea is to consider the wave packet transform of (3.3) with respect to $\psi(t,\cdot)$.

Step 1. First we compute $W_{\psi(t,\cdot)}(i\partial_t v(t,\cdot))(x,\xi)$ and $W_{\psi(t,\cdot)}(\frac{1}{2}\Delta v(t,\cdot))(x,\xi)$. By integration by parts, we obtain

$$\begin{aligned} W_{\psi(t,\cdot)}\left(\frac{1}{2}\Delta v(t,\cdot)\right)(x,\xi) &= \frac{1}{2}\int_{\mathbb{R}^n} \overline{\psi(t,y-x)}\Delta_y v(t,y)e^{-iy\cdot\xi}dy \\ &= W_{\frac{1}{2}\Delta\psi(t,\cdot)}(v(t,\cdot))(x,\xi) + \left(i\xi\cdot\nabla_x - \frac{1}{2}|\xi|^2\right)W_{\psi(t,\cdot)}(v(t,\cdot))(x,\xi) \end{aligned}$$

and

$$\begin{aligned} i\partial_t W_{\psi(t,\cdot)}(v(t,\cdot))(x,\xi) &= i\partial_t\left(\int_{\mathbb{R}^n} \overline{\psi(t,y-x)}v(t,y)e^{-iy\cdot\xi}dy\right) \\ &= W_{-i\partial_t\psi(t,\cdot)}(v(t,\cdot))(x,\xi) + W_{\psi(t,\cdot)}(i\partial_t v(t,\cdot))(x,\xi). \end{aligned}$$

Next we compute $W_{\psi(t,\cdot)}(\widetilde{V}(t,\cdot)v(t,\cdot))(x,\xi)$. For this, we use Taylor's theorem for $\widetilde{V}(t,\cdot)$, i.e.,

$$\widetilde{V}(t,y) = \widetilde{V}(t,x) + \nabla_x\widetilde{V}(t,x)\cdot(y-x) + \sum_{j,k=1}^{n}(y_j-x_j)(y_k-x_k)\widetilde{V}_{j,k}(t,x,y),$$

where

$$\widetilde{V}_{j,k}(t,x,y) = \int_0^1 \partial_{x_j}\partial_{x_k}\widetilde{V}(t,x+\theta(y-x))(1-\theta)d\theta. \tag{3.6}$$

Then by integration by parts, we have

$$\begin{aligned} &W_{\psi(t,\cdot)}(\widetilde{V}(t,\cdot)v(t,\cdot))(x,\xi) \\ &= \left(\widetilde{V}(t,x) + i\nabla_x\widetilde{V}(t,x)\cdot\nabla_\xi - \nabla_x\widetilde{V}(t,x)\cdot x\right)W_{\psi(t,\cdot)}(v(t,\cdot))(x,\xi) \\ &\quad + \underbrace{\sum_{j,k=1}^{n}\int_{\mathbb{R}^n}\overline{\psi(t,y-x)}\widetilde{V}_{jk}(t,x,y)(y_j-x_j)(y_k-x_k)v(t,y)e^{-iy\cdot\xi}dy}_{Rv(x,y,\xi)}. \end{aligned}$$

Step 2. Combining above equalities with

$$W_{[i\partial_t\psi(t,\cdot)+\frac{1}{2}\Delta\psi(t,\cdot)]}(v(t,\cdot))(x,\xi)=0,$$

the initial value problem (3.3) is transformed to

$$\begin{cases}\Big(i\partial_t+i\xi\cdot\nabla_x-i\nabla_x\widetilde{V}(t,x)\cdot\nabla_\xi-\frac{1}{2}|\xi|^2-\widetilde{V}(t,x)\\ \qquad\qquad +\nabla_x\widetilde{V}(t,x)\cdot x\Big)W_{\psi(t,\cdot)}(v(t,\cdot))(x,\xi)-Rv(t,x,\xi)=0,\\ W_{\psi(0,\cdot)}(v(0,\cdot))(x,\xi)=W_{\psi_0}v_0(x,\xi).\end{cases}$$

By the method of characteristics, we obtain

$$W_{\psi(t,\cdot)}(v(t,\cdot))(x,\xi)=e^{-i\int_0^t h(s)ds}\left(W_{\psi_0}v_0(x(0),\xi(0))-i\int_0^t e^{i\int_0^\tau h(s)ds}Rv(\tau,x(\tau),\xi(\tau))d\tau\right),$$

where $x(s)$ and $\xi(s)$ are solutions to

$$\frac{d}{ds}x(s)=\xi(s),\quad \frac{d}{ds}\xi(s)=-(\nabla_x\widetilde{V})(s,x(s))\quad \textit{with}\quad x(t)=x,\quad \xi(t)=\xi.$$

and

$$h(s)=\frac{1}{2}|\xi(s)|^2+\widetilde{V}(s,x(s))-\nabla_x\widetilde{V}(s,x(s))\cdot x(s).$$

Step 3. By taking L^p-norm with respect to x and ξ on both side, we have

$$\begin{aligned}\|v(t,\cdot)\|_{M^{p,p}_{\psi(t,\cdot)}}&=\|W_{\psi(t,\cdot)}(v(t,\cdot))(x,\xi)\|_{L^p_{x,\xi}}\\ &\le\|W_{\psi_0}v_0(x(0),\xi(0))\|_{L^p_{x,\xi}}+\int_0^t\|Rv(\tau,x(\tau),\xi(\tau))\|_{L^p_{x,\xi}}d\tau\\ &=I+II.\end{aligned}$$

To estimate I, we consider the change of variables $(x,\xi)\mapsto(X,\Xi)=(x(0),\xi(0))$. By Lemma 3.3 and the implicit function theorem, we have

$$\det\frac{\partial(x,\xi)}{\partial(X,\Xi)}=1$$

and thus

$$I=\left(\iint_{\mathbb{R}^{2n}}|W_{\psi_0}v_0(X,\Xi)|^p\left|\det\frac{\partial(x,\xi)}{\partial(X,\Xi)}\right|dXd\Xi\right)^{\frac{1}{p}}=\|v_0\|_{M^{p,p}_{\psi_0}}.$$

To estimate II, we use the equality

$$(1-\Delta_y)^N e^{iy\cdot(\eta-\xi(\tau))}=\langle\eta-\xi(\tau)\rangle^{2N}e^{iy\cdot(\eta-\xi(\tau))},\quad N\in\mathbb{N},$$

the inversion formula for wave packet transform

$$v(t,y) = \frac{1}{\|\psi(t,\cdot)\|_{L^2}^2}\left(W^*_{\psi(t,\cdot)}W_{\psi(t,\cdot)}v(t,\cdot)\right)(y),$$

and also the assumption on V. Then by the change of variables $(x,\xi) \mapsto (X,\Xi) = (x(\tau),\xi(\tau))$, Young's inequality and Lemma 3.3 yield

$$II \le C_T \int_0^t \|v(\tau,\cdot)\|_{M^{p,p}_{\psi(\tau,\cdot)}} d\tau.$$

Step 4. From above argument, we have

$$\|v(t,\cdot)\|_{M^{p,p}_{\psi(t,\cdot)}} \le \|v_0\|_{M^{p,p}_{\psi_0}} + C_T \int_0^t \|v(\tau,\cdot)\|_{M^{p,p}_{\psi(\tau,\cdot)}} d\tau$$

for $t \in [0,T]$. Then Gronwall's inequality yields

$$\|u(t-s_0,\cdot)\|_{M^{p,p}_{\varphi(t-s_0,\cdot)}} = \|v(t,\cdot)\|_{M^{p,p}_{\psi(t,\cdot)}} \le C_T\|v_0\|_{M^{p,p}_{\psi_0}} = \|u(-s_0,\cdot)\|_{M^{p,p}_{\varphi(-s_0,\cdot)}}$$

for $t \in [0,T]$.

§ 3.3. Proof of Lemma 3.2

We only consider the case $t \in [0,T]$. We can treat the case $t \in [-T,0]$ in the same way. From Step 2 in the proof of Lemma 3.1, we have

$$\begin{aligned}|W_{\psi(t,\cdot)}(v(t,\cdot))(x,\xi)| &\le |W_{\psi_0}v_0(x(0),\xi(0))| + \int_0^t |Rv(\tau,x(\tau),\xi(\tau))|d\tau \\ &= III + IV.\end{aligned}$$

Step 1. We first consider $M^{\infty,1}$-estimate for $v(t,\cdot) = u(t-s_0,\cdot)$. To estimate III, we use the equality

$$(1-\Delta_y)^N e^{iy\cdot(\eta-\xi(\tau))} = \langle \eta-\xi(\tau)\rangle^{2N} e^{iy\cdot(\eta-\xi(\tau))}, \quad N \in \mathbb{N}$$

and the inversion formula for wave packet transform

$$v_0(y) = \frac{1}{\|\psi_0\|_{L^2}^2}(W^*_{\psi_0}W_{\psi_0}v_0)(y).$$

Then by Lemma 3.4 we have

$$|III| \le C(1+|t|)^{2N} \sum_{|\beta_1|+|\beta_2|\le 2N} \iiint_{\mathbb{R}^{3n}} |\partial_y^{\beta_1}\psi_0(y-x(0))|\,|\partial_y^{\beta_2}\psi_0(y-z)|\frac{|W_{\psi_0}v_0(z,\eta)|}{\langle \xi-\eta\rangle^{2N}}dzd\eta dy$$

and thus for sufficiently large N

$$\begin{aligned}\big\|\|III\|_{L^\infty_x}\big\|_{L^1_\xi} &\le C(1+|t|)^{2N}\sum_{|\beta_1|+|\beta_2|\le 2N}\|\partial_y^{\beta_2}\psi_0\|_{L^1}\\ &\quad\times\left\|\left\|\iint_{\mathbb{R}^{2n}}|\partial_y^{\beta_1}\psi_0(y-x(0))|\frac{\|W_{\psi_0}v_0(z,\eta)\|_{L^\infty_z}}{\langle\eta-\xi\rangle^{2N}}d\eta dy\right\|_{L^\infty_x}\right\|_{L^1_\xi}\\ &\le C_T\|v_0\|_{M^{\infty,1}_{\psi_0}}\end{aligned}$$

for $t\in[0,T]$.

To estimate IV, we use the equality

$$(1-\Delta_y)^N e^{iy\cdot(\eta-\xi(\tau))} = \langle\eta-\xi(\tau)\rangle^{2N}e^{iy\cdot(\eta-\xi(\tau))},\quad N\in\mathbb{N},$$

the inversion formula for wave packet transform

$$v(t,y)=\frac{1}{\|\psi(t,\cdot)\|^2_{L^2}}\left(W^*_{\psi(t,\cdot)}W_{\psi(t,\cdot)}v(t,\cdot)\right)(y)$$

and the assumption on V. Then by Lemma 3.4 we have

$$\begin{aligned}|Rv(\tau,x(\tau),\xi(\tau))| &\le \frac{C(1+|t-\tau|)^{2N}}{\|\psi(\tau,\cdot)\|^2_{L^2}}\\ &\quad\times\sum_{j,k=1}^{n}\sum_{|\beta_1|+|\beta_2|+|\beta_3|\le 2N}\iiint_{\mathbb{R}^{3n}}|\partial_y^{\beta_1}\psi_{jk}(\tau,y-x(\tau))|\\ &\quad\times|\partial_y^{\beta_2}\widetilde{V}_{jk}(\tau,x(\tau),y)\partial_y^{\beta_3}\psi(\tau,y-z)|\frac{|W_{\psi(\tau,\cdot)}(v(\tau,\cdot))(z,\eta)|}{\langle\eta-\xi\rangle^{2N}}dzd\eta dy,\end{aligned}$$

where $\psi_{jk}(t,y)=y_jy_k\psi(t,y)$ and $\widetilde{V}_{jk}$ is defined by (3.6). Thus for sufficiently large N

$$\|\|IV\|_{L^\infty_x}\|_{L^1_\xi}\le C'_T\int_0^t\|v(\tau,\cdot)\|_{M^{\infty,1}_{\psi(\tau,\cdot)}}d\tau$$

for $t\in[0,T]$. Hence, we have

$$\|v(t,\cdot)\|_{M^{\infty,1}_{\psi(t,\cdot)}}\le C_T\|v_0\|_{M^{\infty,1}_{\psi_0}}+C'_T\int_0^t\|v(\tau,\cdot)\|_{M^{\infty,1}_{\psi(\tau,\cdot)}}d\tau \tag{3.7}$$

for $t\in[0,T]$. Applying Gronwall's inequality, we have

$$\|v(t,\cdot)\|_{M^{\infty,1}_{\psi(t,\cdot)}}\le C''_T\|v_0\|_{M^{\infty,1}_{\psi_0}}.$$

Step 2. Next we give $M^{1,\infty}$-estimate for $v(t,\cdot)$. Similarly, by Lemma 3.4 we have

$$\begin{aligned}\|\|III\|_{L^1_x}\|_{L^\infty_\xi} \le C(1+|t|)^{2N} \sum_{|\beta_1|+|\beta_2|\le 2N} \Big\| \iiint_{\mathbb{R}^{3n}} &\|\partial_y^{\beta_1}\psi_0(y-x(0))\|_{L^1_x} \\ &\times |\partial_y^{\beta_2}\psi_0(y-z)| \frac{|W_{\psi_0}v_0(z,\eta)|}{\langle \eta-\xi\rangle^{2N}} dz d\eta dy \Big\|_{L^\infty_\xi} \\ \le C_T \|v_0\|_{M^{1,\infty}_{\psi_0}}\end{aligned}$$

and

$$\|\|IV\|_{L^1_x}\|_{L^\infty_\xi} \le C'_T \int_0^t \|u(\sigma,\cdot)\|_{M^{1,\infty}_{\varphi(\tau,\cdot)}} d\tau$$

for $t\in[0,T]$. Hence

$$\|v(t,\cdot)\|_{M^{1,\infty}_{\psi(t,\cdot)}} \le C_T\|v_0\|_{M^{1,\infty}_{\psi_0}} + C'_T \int_0^t \|v(\sigma,\cdot)\|_{M^{1,\infty}_{\psi(\tau,\cdot)}} d\tau \tag{3.8}$$

for $t\in[0,T]$. Applying Gronwall's inequality, we have

$$\|v(t,\cdot)\|_{M^{1,\infty}_{\psi(t,\cdot)}} \le C''_T \|v_0\|_{M^{1,\infty}_{\psi_0}}.$$

Step 3. Finally, we consider the general case. From Lemma 3.1, we have

$$\|v(t,\cdot)\|_{M^{p,p}_{\psi(t,\cdot)}} \le C_T \|v_0\|_{M^{p,p}_{\psi_0}}.$$

Combing Step 1 and Step 2, we have, by the complex interpolation theorem for modulation space,

$$\|v(t,\cdot)\|_{M^{p,q}_{\psi(t,\cdot)}} \le \widetilde{C}_T \|v_0\|_{M^{p,q}_{\psi_0}}$$

for $t\in[0,T]$. Therefore we obtain the desired result.

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